# The Asymptotics of Strongly Regular Graphs

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**Abstract**

A strongly regular graph is called trivial if it or its complement is a union of disjoint cliques. We prove that the parameters $n, k, \lambda, \mu$ of nontrivial strongly regular graphs satisfy

$$\lambda = k^2/n + o(n) \quad \text{and} \quad \mu = k^2/n + o(n).$$

It follows, in particular, that every infinite family of nontrivial strongly regular graphs is quasi-random in the sense of Chung, Graham and Wilson.

## 1 Introduction

Our graph-theoretic notation is standard (see, e.g. [1]). Given a graph $G$ and a set $R \subset V(G)$, we write $\widehat{d}(R)$ for the number vertices in $G$ joined to every vertex in $R$ and call the value $\widehat{d}(R)$ the *codegree* of $R$.

A *strongly regular graph* (*srg* for short) with parameters $n, k, \lambda, \mu$ is a $k$-regular graph of order $n$ such that $\widehat{d}(uv) = \lambda$ if $uv$ is an edge, and $\widehat{d}(uv) = \mu$ if $uv$ is not an edge; we denote by $SR(n, k, \lambda, \mu)$ a srg with parameters $n, k, \lambda, \mu$.

Observe that any graph $rK_m$ is an $SR(mr, m-1, m-2, 0)$; we call these graphs and their complements *trivial* srgs.

Srgs have been intensively studied; we refer the reader to, e.g. [5], [2], and [4]. Among the many problems related to srgs, probably the most intriguing one is to find strong necessary conditions for the parameters of a srg. Despite the numerous partial results, no exact condition of wide scope is known. If we look for asymptotic conditions, however, the problem becomes more tangible.

In this note we investigate the parameters of nontrivial srgs when the order tends to infinity. Somewhat surprisingly it turns out that the parameters $\lambda$ and $\mu$ are asymptotically equal. More precisely, the following theorem holds.

**Theorem 1** *The parameters $n, k, \lambda, \mu$ of nontrivial strongly regular graphs satisfy*

$$\lambda = k^2/n + o(n) \quad \textit{and} \quad \mu = k^2/n + o(n). \tag{1}$$

In terms of quasi-random graphs (e.g., see [6], [8]) this result implies that every infinite family of nontrivial srgs is quasi-random.

Recently Cameron [3] discussed the randomness aspect of srgs; however, already Thomason [9] suggested that close relations between srgs and quasi-random graphs might exist. Our result shows that, in fact, there is a straightforward relationship.

To prove Theorem 1 we shall use Semerédi's Uniformity Lemma (SUL for short) - a widely applicable tool in extremal graph theory, but seldom, if ever, applied to "rigid" combinatorial objects like srgs.

In Section 2 we give the notions related to SUL and several counting lemmas; the proof of Theorem 1 is presented in Section 3.

## 2 Semerédi's Uniformity Lemma

For expository matter on Szemerédi's uniformity lemma (SUL) the reader is referred to [7] and [1]. This remarkable result is usually called Szemerédi's Regularity Lemma, but the term "uniformity" seems more appropriate to its spirit.

We shall introduce some notation. Given a graph $G$, if $u \in V(G)$ and $Y \subset V(G)$, we write $d_Y(u)$ for the number of neighbors of $u$ in $Y$; similarly, if $R \subset V(G)$, we write $\widehat{d_Y}(R)$ for the number of vertices in $Y$ that are joined to every vertex in $R$. The set of neighbors of a vertex $u$ is denoted by $\Gamma(u)$.

Let $G$ be a graph; if $A, B \subset V(G)$ are nonempty disjoint sets, we write $e(A, B)$ for the number of $A - B$ edges; the value

$$d(A, B) = \frac{e(A, B)}{|A| |B|}$$

is called the *density* of the pair $(A, B)$.

Let $\varepsilon > 0$; a pair $(A, B)$ of two nonempty disjoint sets $A, B \subset V(G)$ is called $\varepsilon$-*uniform* if the inequality

$$|d(A, B) - d(X, Y)| < \varepsilon$$

holds for every $X \subset A$, $Y \subset B$ with $|X| \geq \varepsilon |A|$ and $|Y| \geq \varepsilon |B|$.

We shall use SUL in the following form.

**Theorem 2 (Szemerédi's Uniformity Lemma)** *Let $l \geq 1$, $\varepsilon > 0$. There exists $M = M(\varepsilon, l)$ such that, for every graph $G$ of sufficiently large order $n$, there exists a partition $V(G) = \cup_{i=0}^{p} V_i$ satisfying $l \leq p \leq M$ and:*

(i) $|V_0| < \varepsilon n$, $|V_1| = ... = |V_p|$;

(ii) *for every $i \in [p]$, all but at most $\varepsilon p$ pairs $(V_i, V_j)$, $(j \in [p] \setminus \{i\})$, are $\varepsilon$-uniform.*

Usually SUL is stated with a weaker and less convenient form of condition *(ii)*; the above form, however, is easily implied.

We present below some counting lemmas needed in the proof of the main theorem. Lemmas of this kind are known and their proofs are routine, nevertheless, for the sake of completeness, we present them in some detail.

For every integer $r \geq 0$, set

$$\phi(r) = r! \sum_{i=0}^{r-1} \frac{1}{i!}.$$

**Lemma 3** *Let $\varepsilon > 0$, $r \geq 1$, and $(A, B)$ be an $\varepsilon$-uniform pair with $d(A, B) = d$. If $Y \subset B$ and $(d-\varepsilon)^{r-1}|Y| > \varepsilon|B|$, then fewer than*

$$\varepsilon\phi(r)\binom{|A|}{r}$$

*$r$-sets $R \subset A$ satisfy*

$$d_Y(R) \leq (d-\varepsilon)^r |Y|. \tag{2}$$

**Proof** Since this result is essentially known (see [7], Fact 1.4), we shall only sketch the proof. We use induction on $r$. Let $\mathcal{F}_r$ be the class of $r$-sets in $A$ satisfying inequality (2). Observe that those members of $\mathcal{F}_{r+1}$ that contain a member of $\mathcal{F}_r$ are at most $|\mathcal{F}_r|(|A|-r)n$; also, for every $r$-set $R \notin \mathcal{F}_r$, at most $\varepsilon|A|$ members of $\mathcal{F}_{r+1}$ contain $R$. Therefore,

$$|\mathcal{F}_{r+1}| \leq \varepsilon|A|\binom{|A|-1}{r} + |\mathcal{F}|(|A|-r)$$

and the assertion follows. $\square$

With a simple change of signs we obtain a twin result.

**Lemma 4** *Let $\varepsilon > 0$, $r \geq 1$, and $(A, B)$ be an $\varepsilon$-uniform pair with $d(A, B) = d$. If $Y \subset B$ and $(d+\varepsilon)^{r-1}|Y| > \varepsilon|B|$, then fewer than*

$$\varepsilon\phi(r)\binom{|A|}{r}$$

*$r$-sets $R \subset A$ satisfy*

$$d_Y(R) \geq (d+\varepsilon)^r |Y|. \tag{3}$$

Lemmas 3 and 4 imply the following statement.

**Lemma 5** *Let $\varepsilon > 0$, $r \geq 1$, and $(A, B)$ be an $\varepsilon$-uniform pair with $d(A, B) = d$. Then:*

(i) *at least*

$$(1-\varepsilon\phi(r))\binom{|A|}{r}$$

*$r$-sets $R \subset A$ satisfy*

$$\widehat{d}_B(R) - d^r|B| > -\varepsilon r|B|;$$

(ii) *at least*

$$\left(1-\varepsilon\phi\left(r\right)\right)\binom{|A|}{r}$$

*$r$-sets $R\subset A$ satisfy*

$$\widehat{d}_B\left(R\right)-d^r\left|B\right|<\varepsilon r\left|B\right|.$$

**Proof** To prove assertion *(i)*, observe first that it holds trivially if $d^r<\varepsilon r$. On the other hand, $d^r\geq\varepsilon r$ implies $\left(d-\varepsilon\right)^{r-1}\geq\varepsilon$; applying Lemma 4 with $Y=B$, we deduce that at least

$$\left(1-\varepsilon\phi\left(r\right)\right)\binom{|A|}{r}$$

$r$-sets $R\subset A$ satisfy

$$d_B\left(R\right)>\left(d-\varepsilon\right)^r\left|B\right|>\left(d^r-r\varepsilon d^{r-1}\right)\left|B\right|\geq\left(d^r-r\varepsilon\right)\left|B\right|,$$

completing the proof of *(i)*.

To prove assertion *(ii)*, we use induction on $r$. For $r=1$ the assertion follows from Lemma 4 with $Y=B$; assume $r\geq 2$ and the assertion true for $r'<r$.

Observe that, if $\varepsilon>1-d$, we deduce

$$d^r+\varepsilon r>d^r+\left(1-d\right)r\geq 1,$$

and the assertion follows trivially. From $\varepsilon\leq 1-d$ we find that

$$d^r+\varepsilon r\geq d^r+\varepsilon\left(\left(d+\varepsilon\right)^{r-1}+\varepsilon\left(d+\varepsilon\right)^{r-2}+...+\varepsilon^{r-1}\right)=\left(d+\varepsilon\right)^r$$

so, provided $\left(d+\varepsilon\right)^{r-1}\geq\varepsilon$ holds, we may apply Lemma 4 with $Y=B$ and complete the proof of *(ii)*.

It remains to consider the case $\left(d+\varepsilon\right)^{r-1}<\varepsilon$ which is only possible if $r>2$. Let $\mathcal{F}$ be the family of all $(r-1)$-sets $R\subset A$ satisfying

$$\widehat{d}_B\left(R\right)-d^{r-1}\left|B\right|<\varepsilon\left(r-1\right)\left|B\right|;$$

by the inductive assumption,

$$\left|\mathcal{F}\right|>\left(1-\varepsilon\phi\left(r-1\right)\right)\binom{|A|}{r-1}.$$

If an $r$-set $R\subset A$ contains a member $R'\in\mathcal{F}$, we find that

$$\widehat{d}_B\left(R\right)\leq\widehat{d}_B\left(R'\right)<\left(\varepsilon\left(r-1\right)+d^{r-1}\right)\left|B\right|<\varepsilon r\left|B\right|\leq\left(\varepsilon r+d^r\right)\left|B\right|.$$

Since there are at least

$$\frac{\left|\mathcal{F}\right|\left(n-r+1\right)}{r}>\left(1-\varepsilon\phi\left(r-1\right)\right)\binom{|A|}{r}>\left(1-\varepsilon\phi\left(r\right)\right)\binom{|A|}{r}$$

such $r$-sets, the proof is completed. □

Next we shall present a similar result for pairs across different vertex classes.

**Lemma 6** *Let* $\varepsilon > 0$ *and* $(A_1, B)$ , $(A_2, B)$ *be* $\varepsilon$*-uniform pairs with* $d(A_1, B) = d_1$ *and* $d(A_2, B) = d_2$. *Then:*

(i) *at least* $(1 - 2\varepsilon)|A_1||A_2|$ *pairs* $(u, v) \in A_1 \times A_2$ *satisfy*

$$\widehat{d}_B(uv) - d_1 d_2 |B| > -2\varepsilon |B|;$$

(ii) *at least* $(1 - 2\varepsilon)|A_1||A_2|$ *pairs* $(u, v) \in A_1 \times A_2$ *satisfy*

$$\widehat{d}_B(uv) - d_1 d_2 |B| < 2\varepsilon |B|.$$

**Proof** To prove assertion *(i)*, observe first that it holds trivially if $d_1 < 2\varepsilon$ or $d_2 < 2\varepsilon$, so we shall assume $d_1 \geq 2\varepsilon$ and $d_2 \geq 2\varepsilon$. Let

$$X = \{u \in A_1 : d_B(u) \leq (d_1 - \varepsilon)|B|\}.$$

Applying Lemma 3 to the pair $(A_1, B)$ with $r = 1$, $Y = B$, we find that $|X| < \varepsilon |A_1|$. Select any $u \in A_1 \backslash X$, and let

$$Y = \left\{v \in A_2 : \widehat{d}_B(uv) \leq (d_2 - \varepsilon) d_B(u)\right\}.$$

Applying Lemma 3 to the pair $(A_2, B)$ with $r = 1$ and $Y = \Gamma(u) \cap B$ , we find that $|Y| < \varepsilon |A_2|$. Therefore, at least

$$(1 - \varepsilon)^2 |A_1||A_2| > (1 - 2\varepsilon)|A_1||A_2|$$

pairs $(u, v) \in A_1 \times A_2$ satisfy

$$\widehat{d}_B(uv) > (d_1 - \varepsilon)(d_2 - \varepsilon)|B| > d_1 d_2 |B| - 2\varepsilon |B|,$$

completing the proof of *(i)*.

To prove assertion *(ii)*, observe first that, if

$$(d_1 + \varepsilon)(d_2 + \varepsilon) > d_1 d_2 + 2\varepsilon, \tag{4}$$

we deduce

$$d_1 d_2 + 2\varepsilon > 4 - 2d_1 - 2d_2 + d_1 d_2 \geq (2 - d_1)(2 - d_2) \geq 1,$$

and the assertion follows trivially, so we shall assume that (4) fails. Applying the same argument as in the proof of *(i)*, we find that at least $(1 - 2\varepsilon)|A_1||A_2|$ pairs $(u, v) \in A_1 \times A_2$ satisfy the inequality

$$\widehat{d}_B(uv) < (d_1 + \varepsilon)(d_2 + \varepsilon)|B| \leq (d_1 d_2 + 2\varepsilon)|B|,$$

completing the proof of *(ii)*. $\square$

## 2.1 Sums and averages of codegrees

In this subsection we shall investigate codegrees in graphs consisting of several $\varepsilon$-uniform pairs.

**Lemma 7** *Let $\varepsilon > 0$ and $H$ be a graph whose vertices are partitioned as*

$$V(H) = A \cup B_1 \cup ... \cup B_p$$

*so that*

$$|A| = |B_1| = ... = |B_p| = t.$$

*For every $i \in [p]$, let the pair $(A, B_i)$ be $\varepsilon$-uniform and set $d(A, B_i) = d_i$. Then the inequality*

$$\left| \sum_{\{u,v\} \in S} \sum_{i=1}^{p} \widehat{d}_{B_i}(uv) - t|S| \sum_{i=1}^{p} d_i^2 \right| < 5p\varepsilon t^3$$

*holds for every set $S$ of 2-sets in $A$.*

**Proof** We shall prove first that, for every $i \in [p]$,

$$-5\varepsilon t^3 \le \sum_{\{u,v\} \in S} \widehat{d}_{B_i}(uv) - t|S| d_i^2 \le 5\varepsilon t^3. \tag{5}$$

Indeed, applying Lemma 5 to the pair $(A, B_i)$ with $r = 2$ and $Y = B_i$, we find that at least $|S| - 4\varepsilon t^2$ sets $\{u, v\} \in S$ satisfy

$$-2\varepsilon t < \widehat{d}_{B_i}(uv) - d_i^2 t < 2\varepsilon t,$$

and, therefore,

$$-2\varepsilon t|S| - 4\varepsilon t^3 < \sum_{\{u,v\} \in S} \widehat{d}_{B_i}(uv) - d_i^2 t|S| < 2\varepsilon t|S| + 4\varepsilon t^3.$$

As $|S| < t^2/2$, inequality (5) follows; summing it for $i = 1, ..., p$ we obtain the desired result. □

**Corollary 8** *Under the conditions of Lemma 7, if $|S| \ge \alpha t^2$ for some $\alpha > 0$, then,*

$$\left| \frac{1}{|S|} \sum_{\{u,v\} \in S} \sum_{i=1}^{p} \widehat{d}_{B_i}(uv) - t \sum_{i=1}^{p} d_i^2 \right| < \frac{5p\varepsilon}{\alpha} t.$$

**Lemma 9** *Suppose $\varepsilon > 0$ and $H$ is a graph whose vertices are partitioned as*

$$V(H) = A_1 \cup A_2 \cup B_1 \cup ... \cup B_p$$

*so that*

$$|A_1| = |A_2| = |B_1| = ... = |B_k| = t.$$

*For every $i \in [2]$, $j \in [k]$, let the pair $(A_i, B_j)$ be $\varepsilon$-uniform and set $d(A_i, B_j) = d_{ij}$. Then, the inequality*

$$\left| \sum_{(u,v)\in S} \sum_{i=1}^{p} \widehat{d}_{B_i}(uv) - t|S| \sum_{i=1}^{k} d_{1i}d_{2i} \right| < 6\varepsilon p t^3$$

*holds for every set $S \subset A_1 \times A_2$.*

**Proof** We shall prove first that, for every $i \in [p]$,

$$-6\varepsilon t^3 \leq \sum_{(u,v)\in S} \widehat{d}_{B_i}(uv) - t|S| d_{1i}d_{2i} \leq 6\varepsilon t^3. \tag{6}$$

Indeed, applying Lemma 6 with $B = B_i$, we find that at least $|S| - 4\varepsilon t^2$ pairs $(u, v) \in S$ satisfy

$$-2\varepsilon t < \widehat{d}_{B_i}(uv) - d_{1i}d_{2i}t < 2\varepsilon t,$$

and, therefore,

$$-2\varepsilon t|S| - 4\varepsilon t^3 < \sum_{(u,v)\in S} \widehat{d}_{B_i}(uv) - d_{1i}d_{2i}t|S| < 2\varepsilon t|S| + 4\varepsilon t^3.$$

As $|S| \leq t^2$, inequality (6) follows; summing it for $i = 1, ..., p$ we obtain the desired result. □

**Corollary 10** *Under the conditions of Lemma 9, if $|S| \geq \alpha t^2$ for some $\alpha > 0$, then,*

$$\left| \frac{1}{|S|} \sum_{(u,v)\in S} \sum_{i=1}^{p} \widehat{d}_{B_i}(uv) - t \sum_{i=1}^{p} d_{1i}d_{2i} \right| < \frac{6p\varepsilon}{\alpha} t.$$

# 3 Proof of the main theorem

Let $d \geq 0$, $a \geq 0$, $c \geq 0$. A sequence $\{SR(n_s, k_s, \lambda_s, \mu_s)\}_{s=1}^{\infty}$ of srgs of increasing order such that

$$\lim_{s\to\infty} \frac{k_s}{n_s} = d, \quad \lim_{s\to\infty} \frac{\lambda_s}{n_s} = a, \quad \lim_{s\to\infty} \frac{\mu_s}{n_s} = c$$

is called a $CSR(d, a, c)$ *sequence.*

Note that to prove Theorem 1 it suffices to show that the parameters $d, a, c$ of any $CSR(d, a, c)$ sequence of nontrivial srgs satisfy the equalities

$$a = c = d^2. \tag{7}$$

Indeed, assume Theorem 1 false - that is to say, there exist $\varepsilon > 0$ and a sequence $\{SR(n_s, k_s, \lambda_s, \mu_s)\}_{s=1}^{\infty}$ of nontrivial srgs of increasing order such that

$$\left|\frac{\lambda_s}{n_s} - \frac{k_s^2}{n_s^2}\right| > \varepsilon \quad \text{or} \quad \left|\frac{\mu_s}{n_s} - \frac{k_s^2}{n_s^2}\right| > \varepsilon. \tag{8}$$

From the sequence $\{SR(n_s, k_s, \lambda_s, \mu_s)\}_{s=1}^{\infty}$ we can always select a subsequence that is a $CSR(d, a, c)$ sequence for some $d \geq 0$, $a \geq 0$, $c \geq 0$; in view of inequalities (8), condition (7) fails, as claimed.

To prove equalities (7) we shall establish some facts about $CSR(d, a, c)$ sequences. Observe first that, if $\{G_s\}_{s=1}^{\infty}$ is a $CSR(d, a, c)$ sequence, then $\{\overline{G_s}\}_{s=1}^{\infty}$ is a

$$CSR(1 - d, 1 - 2d + c, 1 - 2d + a)$$

sequence.

Also, the well-known relations

$$k > \lambda, \quad k \geq \mu, \quad k(k - \lambda - 1) = (n - k - 1)\mu,$$

holding for any $SR(n, k, \lambda, \mu)$, imply that the parameters of any $CSR(d, a, c)$ sequence satisfy

$$d \geq a, \quad d \geq c, \tag{9}$$

$$d^2 - (a - c)d - c = 0. \tag{10}$$

Thus, equalities (7) hold for $d = 0$, and, applying the same argument to $\{\overline{G_s}\}_{s=1}^{\infty}$, they hold for $d = 1$ as well. Therefore, we may and shall assume that $0 < d < 1$.

**Lemma 11** *If $0 < d < 1$ and $\{G_s\}_{s=1}^{\infty}$ is a $CSR(d, a, c)$ sequence of nontrivial srgs then $d \neq a$ and $d \neq c$.*

**Proof** Assume $d = a$; then equality (10) implies $c = 0$. We shall show that $p = d^{-1}$ is integer and for $s$ sufficiently large, $G_s$ is a union of $p$ complete graphs of equal order.

Let $n_s, k_s, \lambda_s, \mu_s$ be the parameters of $G_s$ for $s = 1, 2, ....$ Select any $u \in V(G_s)$ and let $\Gamma(u)$ be the set of its neighbors. Clearly, $|\Gamma(u)| = k_s$ and the graph $G_s[\Gamma(u)]$ is $\lambda_s$-regular. If $v, w \in \Gamma(u)$ are two nonadjacent vertices, then, by the inclusion-exclusion formula, we find that

$$\widehat{d}_{\Gamma(u)}(vw) \geq 2\lambda_s - k_s = k_s + o(n_s),$$

and hence $c = d$. Thus $d = 0$, a contradiction. We conclude that $G[\Gamma(u)]$ is a complete graph of order $k_s$.

Furthermore, $\Gamma(u) \cap \Gamma(v) = \varnothing$ for any two nonadjacent vertices $u, v \in V(G_s)$. Indeed, if $w \in \Gamma(u) \cap \Gamma(v)$, then $u, v \in \Gamma(w)$, and, therefore, must be adjacent, contrary to our choice. Thus for any $u \in V(G_s)$, the set $\Gamma(u_i) \cup \{u_i\}$ is a complete graph of order $k_s + 1$.

Select a maximal independent set $\{u_1, ..., u_p\}$ in $G_s$. Since $\{u_1, ..., u_p\}$ is maximal, we have

$$\cup_{i=1}^{p} (\Gamma(u_i) \cup \{u_i\}) = V(G_s).$$

Thus $d = 1/p$ and $V(G_s)$ is partitioned in $p$ complete graphs of order $k_s + 1$. To complete the proof we have to show that no edge joins vertices from different complete graphs.

Let $uv$ be an edge such that $u \in \Gamma(u_i) \cup \{u_i\}$, $v \in \Gamma(u_j) \cup \{u_j\}$, and $i \neq j$. Since $\Gamma(v)$ is a complete graph and $u \in \Gamma(v)$, then $u$ is adjacent to all vertices of $\Gamma(u_j) \cup \{u_j\}$, implying $d(u) \geq 2k_s + 1$, a contradiction, completing the proof.

The case $d = c$ follows by applying the above argument to the sequence $\{\overline{G_s}\}_{s=1}^{\infty}$. $\square$

**Proof of Theorem 1** Let $\{G_s\}_{s=1}^{\infty}$ be a $CSR(d, a, c)$ sequence of nontrivial srgs and suppose $n_s, k_s, \lambda_s, \mu_s$ are the parameters of $G_s$ for $s = 1, 2, ....$ Our goal is to prove equalities (7). Note that, it suffices to prove that $a = c$, for, then, the equality $a = d^2$ follows immediately from equality (10). Observe that since $G_s$ are nontrivial, by Lemma 11 we have

$$d \neq a, \quad d \neq c.$$

Assume

$$a \neq c,$$

set

$$\delta = \min\left\{|a - c|, |d - a|, |d - c|, \frac{1}{10}\right\}, \tag{11}$$

and let

$$\varepsilon = \left(\frac{\delta}{20}\right)^2,$$
$$l = \lceil 1/\varepsilon \rceil.$$

Select $s$ so large that the inequalities

$$|k_s - dn_s| < \varepsilon n_s, \tag{12}$$
$$|\lambda_s - an_s| < \varepsilon n_s, \tag{13}$$
$$|\mu_s - cn_s| < \varepsilon n_s$$

hold and, in addition, $n_s$ is large enough to apply SUL to $G_s$ with parameters $\varepsilon$ and $l$; for technical reasons we also require that $n_s > 3M(\varepsilon, l)$.

Thus there is a partition $V(G_s) = \cup_{i=0}^{p} V_i$ such that $l \leq p \leq M(\varepsilon, l)$ and:

*i)* $|V_0| < \varepsilon |G_s|$, $|V_1| = ... = |V_p|$;

*ii)* for every $i \in [p]$, all but at most $\varepsilon p$ pairs $(V_i, V_j)$, $(j \in [p] \setminus \{i\})$, are $\varepsilon$-uniform.

Let $n = n_s$, $t = |V_1|$, and set $d_{ij} = d(V_i, V_j)$ for every $i, j \in [p]$, $(i \neq j)$. Observe that the inequality $n > 3M(\varepsilon, l)$ and condition *(i)* imply

$$2 \leq t \leq \frac{n}{p} \leq \frac{n}{l} \leq \varepsilon n \tag{14}$$

and

$$tp \leq n \leq \frac{tp}{1-\varepsilon} < tp\,(1+2\varepsilon)\,. \tag{15}$$

Our first goal is to prove that, if the inequalities

$$\sqrt{\varepsilon}t^2 < e\,(V_i, V_j) < \left(1-\sqrt{\varepsilon}\right) t^2 \tag{16}$$

hold for some pair $(V_i, V_j)$, then the inequality

$$|a-c| < \delta, \tag{17}$$

holds, contradicting the choice of $\delta$.

Suppose a pair $(V_i, V_j)$ satisfies inequalities (16). Let

$$R = \{r : r \in [p] \setminus \{i,j\}\,,\ \ (V_i, V_r) \text{ and } (V_j, V_r) \text{ are } \varepsilon\text{-uniform}\}\,.$$

Observe first that condition *(ii)* implies $|R| \geq (1-2\varepsilon)\,p$. Select any vertex $u \in V_i$; inequality (12) implies

$$(d-\varepsilon)\,n < d\,(u) < (d+\varepsilon)\,n,$$

and, therefore,

$$(d-\varepsilon)\,n < \sum_{r=0}^{p} d_{V_r}\,(u) < (d+\varepsilon)\,n.$$

Hence, in view of $|R| \geq (1-2\varepsilon)\,p$ and $pt \leq n$, we deduce

$$(d-4\varepsilon)\,n < (d-\varepsilon)\,n - 2\varepsilon pt < \sum_{r\in R} d_{V_r}\,(u) < (d+\varepsilon)\,n,$$

and, by inequalities (15), it follows that

$$(d-4\varepsilon)\,pt < \sum_{r\in R} d_{V_r}\,(u) < (d+\varepsilon)\,(1+2\varepsilon)\,pt \leq (d+4\varepsilon)\,pt.$$

Summing this inequality for all $u \in V_i$ and dividing by $t^2$, we obtain

$$(d-4\varepsilon)\,p < \sum_{r\in R} d_{ir} < (d+4\varepsilon)\,p; \tag{18}$$

by symmetry we also have

$$(d-4\varepsilon)\,p < \sum_{r\in R} d_{jr} < (d+4\varepsilon)\,p. \tag{19}$$

Applying Corollary 10 with $A_1 = V_i$, $A_2 = V_j$, $B_r = V_r$ for all $r \in R$, and $S = E\,(V_1, V_2)$, we see that

$$\left| \frac{1}{e\,(V_i, V_j)} \sum_{(u,v)\in E(V_1,V_2)} \frac{1}{t} \sum_{r\in R} \widehat{d}_{V_r}\,(uv) - \sum_{r\in R} d_{1r} d_{2r} \right| < \frac{6p\varepsilon}{\sqrt{\varepsilon}} = 6\sqrt{\varepsilon}p. \tag{20}$$

Furthermore, select any edge $uv$ such that $u \in V_i$ and $v \in V_j$. Condition (13) implies

$$(a-\varepsilon)\, n < \widehat{d}\,(uv) < (a+\varepsilon)\, n;$$

conditions *(i)* and *(ii)* imply

$$0 \leq \widehat{d}\,(uv) - \sum_{r \in R} \widehat{d}_{V_r}\,(uv) = \widehat{d}_{V_0}\,(uv) + \sum_{r \in [p] \setminus R} \widehat{d}_{V_r}\,(uv) < \varepsilon n + 2\varepsilon pt < 3\varepsilon n.$$

It follows that

$$(a-4\varepsilon)\, n < \sum_{r \in R} \widehat{d}_{V_r}\,(uv) < (a+\varepsilon)\, n$$

and, estimating $n$ from (15), we see that

$$(a-4\varepsilon)\, p < \frac{1}{t} \sum_{r \in R} \widehat{d}_{V_r}\,(uv) < (a+\varepsilon)\,(1+2\varepsilon)\, p < (a+4\varepsilon)\, p.$$

Hence, inequality (20) implies

$$\left(a - 10\sqrt{\varepsilon}\right) p < \sum_{r \in R} d_{1r} d_{2r} < \left(a + 10\sqrt{\varepsilon}\right) p. \tag{21}$$

Applying the same argument to any pair $(u, v) \in V_i \times V_j$ such that $uv \notin E\,(V_i, V_j)$, we obtain

$$\left(c - 10\sqrt{\varepsilon}\right) p < \sum_{r \in R} d_{1r} d_{2r} < \left(c + 10\sqrt{\varepsilon}\right) p.$$

These inequalities together with inequalities (21) imply

$$|a - c| < 20\sqrt{\varepsilon} \leq \delta,$$

as claimed.

Therefore, we may and shall assume that condition (16) fails for all pairs $(V_i, V_j)$ - that is to say, for every $i, j \in [p]\,,\ (i \neq j)\,,$ either

$$d_{ij} \leq \sqrt{\varepsilon} \quad \text{or} \quad d_{ij} \geq 1 - \sqrt{\varepsilon}.$$

A simple calculation shows that then

$$0 \leq d_{ij} - d_{ij}^2 \leq \sqrt{\varepsilon} \tag{22}$$

holds for every $i, j \in [p]\,,\ (i \neq j)\,.$ We shall prove that these inequalities imply either

$$|d - a| < \delta \quad \text{or} \quad |d - c| < \delta,$$

contradicting (11).

Assume $e\,(V_1) \geq t^2/5$ and let

$$R = \left\{ r : r \in [p] \setminus \{i\}\,,\ \text{the pair } (V_1, V_r) \text{ is } \varepsilon\text{-uniform} \right\}.$$

As above we establish

$$(d-4\varepsilon)\,p<\sum_{r\in R}d_{1r}<(d+4\varepsilon)\,p;$$

Hence, in view of (22), we obtain

$$\left(d-5\sqrt{\varepsilon}\right)p<\sum_{r\in R}d_{1r}^{2}<\left(d+5\sqrt{\varepsilon}\right)p. \tag{23}$$

Applying Corollary 8 with $A=V_1$, $B_r=V_r$ for all $r\in R$, and $S=E\left(V_1\right)$, we see that

$$\left|\frac{1}{e\left(V_1\right)}\sum_{uv\in E(V_1)}\frac{1}{t}\sum_{r\in R}\widehat{d}_{V_r}\left(uv\right)-\sum_{r\in R}d_{1r}^{2}\right|<\frac{5p\varepsilon}{1/5}=25\varepsilon p.$$

For any edge $uv$ induced by $V_1$, as above, we establish that

$$(a-4\varepsilon)\,pt<\sum_{r\in R}\widehat{d}_{V_r}\left(uv\right)<(a+\varepsilon)\left(1+2\varepsilon\right)pt<(a+4\varepsilon)\,pt.$$

Hence, inequality (23) implies

$$|d-a|<29\varepsilon+5\sqrt{\varepsilon}<\delta,$$

as claimed.

Assuming $e\left(V_1\right)<t^2/5$, from $t\geq 2$, we see that the graph $\overline{G\left[V_1\right]}$ induces at least $t^2/5$ edges. Applying Corollary 8 with $A=V_1$, $B_r=V_r$ for all $r\in R$, and $S=E\left(\overline{G\left[V_1\right]}\right)$, by the above argument applied to the members of $S$, we see that

$$|d-c|<\delta,$$

as claimed. The proof is completed. $\square$

# 4 Concluding remark

Curiously enough, in the proof of Theorem 1 we did not make much use of the essential feature of SUL - the independence of $M\left(\varepsilon,l\right)$ on $n$. This fact suggests that a more involved approach exists, possibly leading to effective bounds on the values

$$\left|\lambda-k^2/n\right| \quad\text{and}\quad \left|\mu-k^2/n\right|.$$

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